

Do They Really Understand?

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So what is  
understanding?

- In math, we have assumed students understand if they get correct answers.
- But is that really true?

If I asked you....

- Why is  $4 + 0 = 4$ ?
  - Why is  $2 - (-8) = 10$ ?
  - Why is  $3/8 = 3 \div 8$ ?
  - Why is  $2/5 \div 1/2 = 4/5$ ?
- 
- Many of us can recite a rule, but do we understand why?

Why do we care?

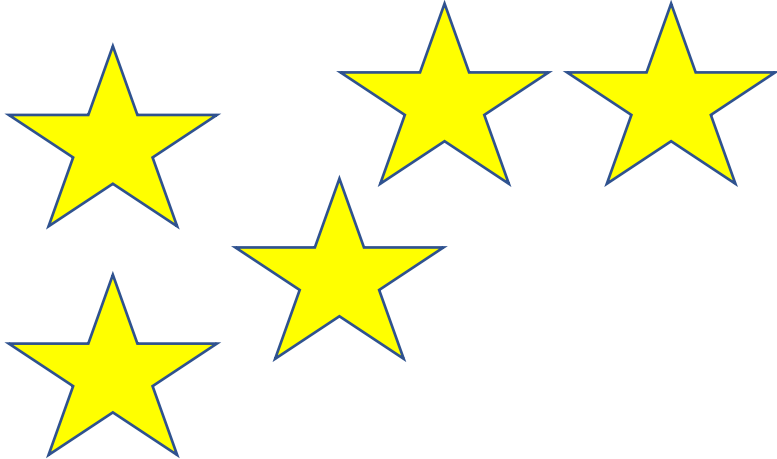
- Retention
- Because shouldn't that be the goal of school anyway?

# Agenda

- We will look at topics at different grade levels, K - 12, to see the difference between a focus on understanding and a focus on just learning procedures.



K-3



- DO IT:
- How many are here?
  
- UNDERSTAND IT:.
- Which is more? Why?

K-3

- DO IT:
- Show 6.

K-3

- UNDERSTAND IT:
- This shows 7.



- How does knowing that help you show 6? 8?



K-3

- DO IT:
- Sarah has 5 stickers. Lia has 2. Who has more?
  
- UNDERSTAND IT:
- Sarah has more stickers than Lia. Kevin has more stickers than Sarah. Who has the most stickers?

K-3

- Sarah has more stickers than Lia. Kevin has more stickers than Sarah. Who has the most stickers?

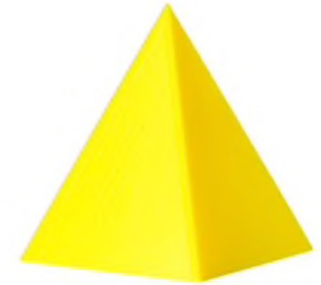
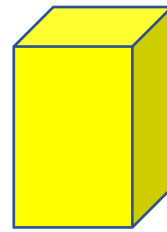
L

S

K

Grades 4 - 6

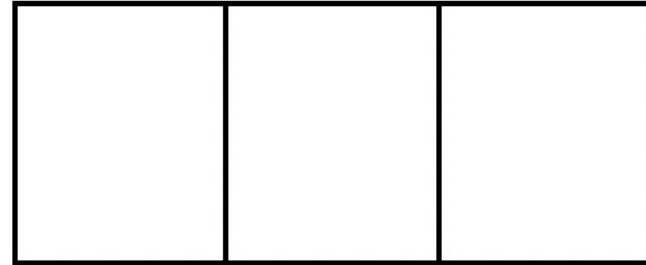
- DO IT:
- Which is a prism?



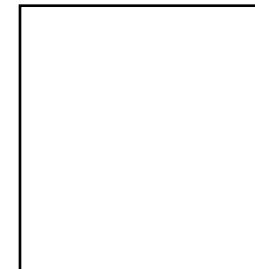
- UNDERSTAND IT:
- A prism and pyramid have the same base. Which has more edges? How do you know?

Grades 4 - 6

- DO IT:
- Show  $\frac{2}{3}$  of this.



- UNDERSTAND IT:
- This is  $\frac{1}{3}$  of something.  
What could the something be?

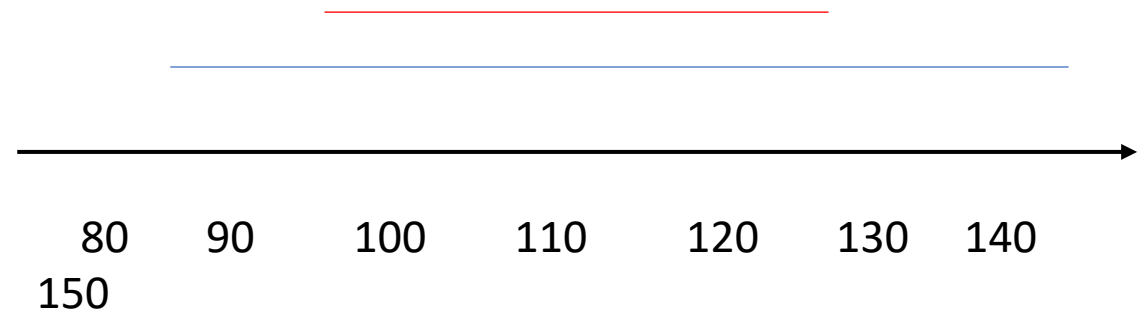


## Grades 4 - 6

- DO IT:
- What is  $150 - 87$ ?
  
- UNDERSTAND IT:
- WITHOUT GETTING THE ANSWERS, how do you know that  $150 - 87$  has to be more than  $131 - 93$ ?

Grades 4 - 6

- How do you know that  $150 - 87$  has to be more than  $131 - 93$ ?

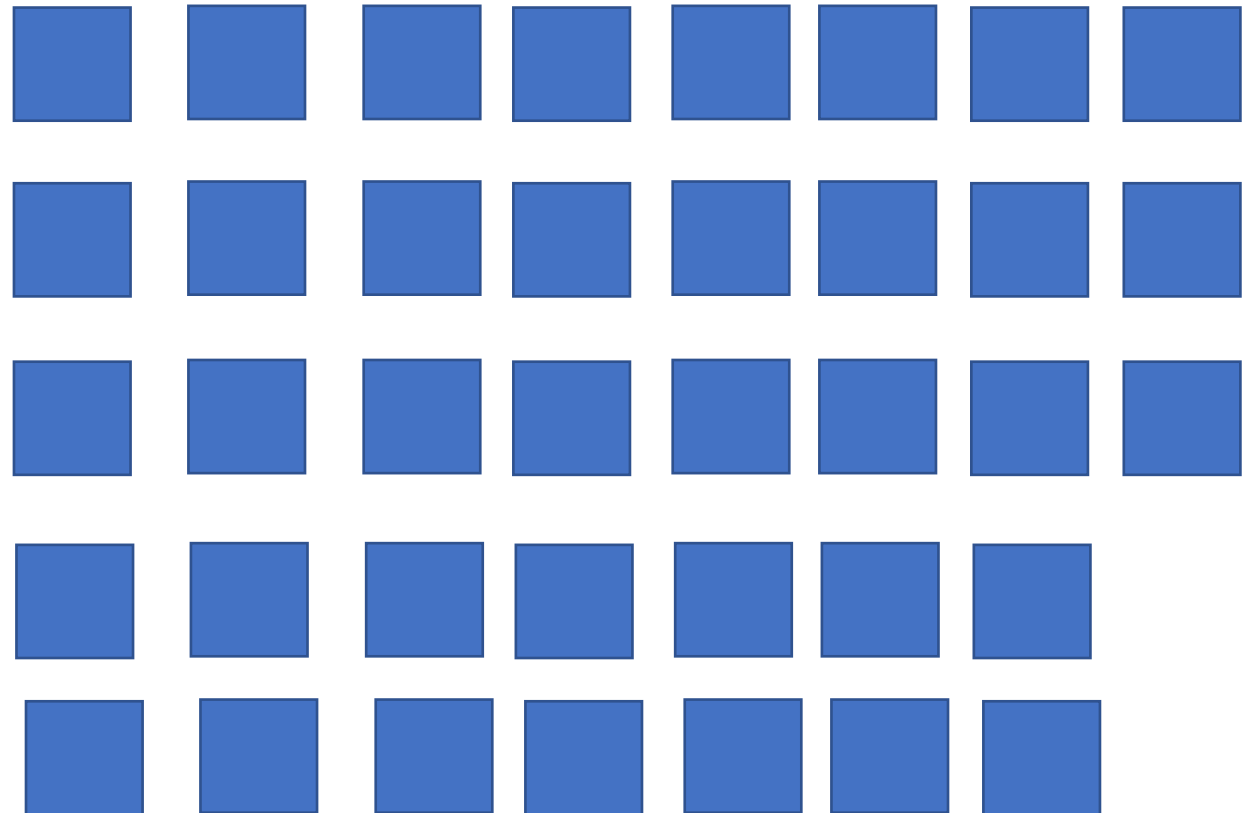


## Grades 4 - 6

- DO IT:
- What is  $4 \times 8$ ?
  
- UNDERSTAND IT:
- How do you know that  $5 \times 7 > 3 \times 8$  without using the answers?

Grades 4 - 6

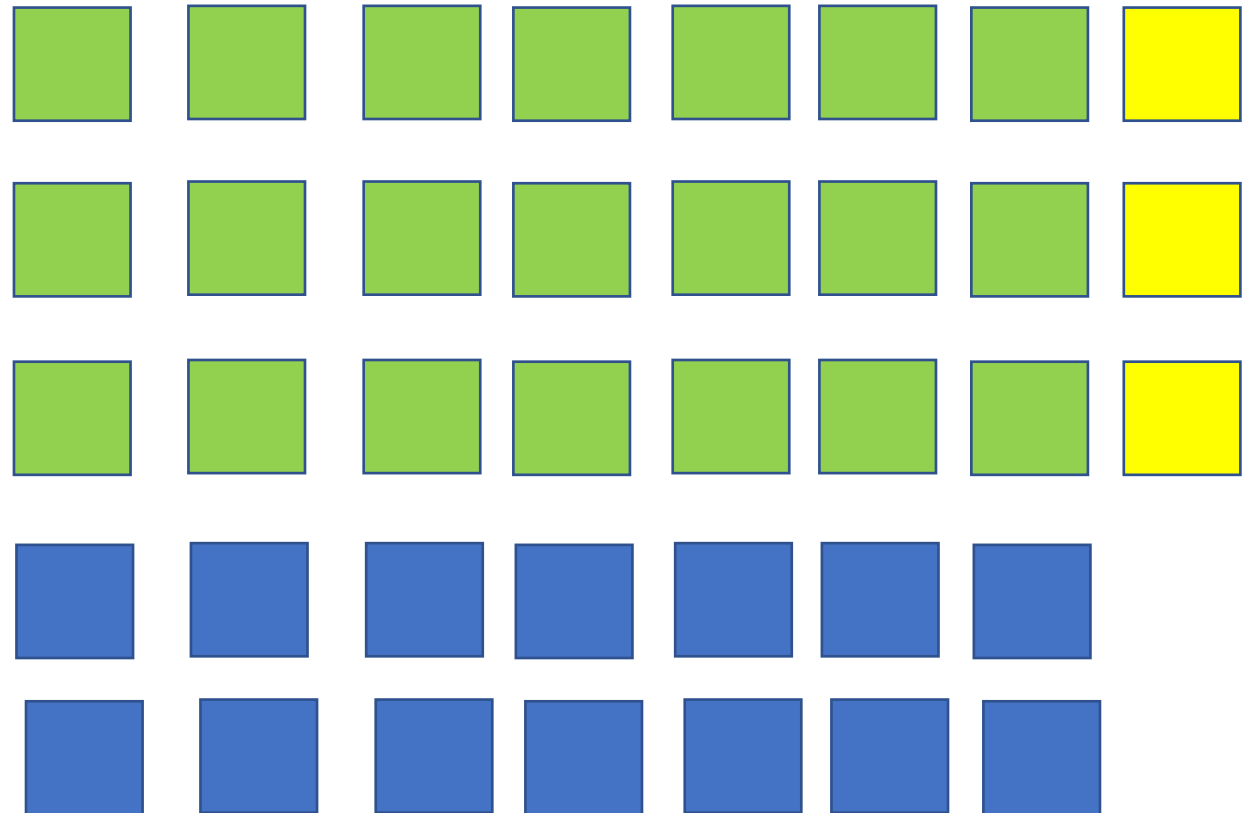
- How do you know that  $5 \times 7 > 3 \times 8$  without using the answers?





Grades 4 - 6

- How do you know that  $5 \times 7 > 3 \times 8$  without using the answers?



Grades 7-9

- You want them to know and use formulas for circumference and area, but what else?

## Grades 7-9

- DO IT:
- What is the circumference of a circle if the diameter is 10 cm?
  
- UNDERSTAND IT:
- Can you know the area of a circle if all you know about it is its circumference?

## Grades 7-9

- DO IT:
  - What is the area of a circle if the radius is 20 cm?
- UNDERSTAND IT:
  - Draw a picture that shows why the formula  $A = \pi r^2$  makes sense and explain how the picture does that.

## Grades 7-9

- DO IT:
- Estimate  $\sqrt{57}$ .
  
- UNDERSTAND IT:
- Draw a picture to show why  $\sqrt{57}$  is between 7 and 8.

## Grades 7-9

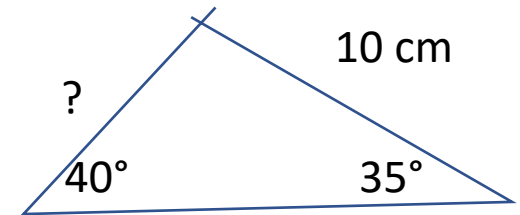
- DO IT: Estimate  $\sqrt{512}$ .
- UNDERSTAND IT: How close together might the square roots of numbers that are 1000 apart be?
- E.g.  $\sqrt{1001}$  and  $\sqrt{1}$  (30)
- $\sqrt{1\,000\,000}$  and  $\sqrt{1001\,000}$  (0.5)

## Grades 9-12

- DO IT: Estimate  $\sqrt{\frac{5}{8}}$ .
- UNDERSTAND IT: The square root of a number is more than the number. What might the number be?

## Grades 9-12

- DO IT: What is the missing side length?



- UNDERSTAND IT: Suppose you know two angles and one side length in a triangle. Can you always figure out the other side lengths?



## Grades 9-12

- DO IT: A set of scores is 22, 99, 102, 33, 59, 78, 102, 81, 65 and 29. Calculate the standard deviation.
- UNDERSTAND IT: The standard deviation of a set of data is quite small. What might the data look like?

So

- You may have noticed question styles, e.g.
- WITHOUT SAYING THE ANSWER
- DRAW A PICTURE TO SHOW
- HOW DOES KNOWING.... TELL YOU...

- Change one of these questions to an understanding one.

Let's do a first try

- What is  $42 - 8$ ?
- What is  $22 \times 23$ ?

- What is  $\frac{7}{3} \times \frac{3}{5}$ ?
- What line goes through  $(2, 3)$  and  $(5, -7)$ ?

Possibles for  $42 - 8$ ?

- When might you subtract a 1-digit number from a 2-digit and get a 1-digit answer?
- You subtract a 1-digit from a 2-digit number. When does the tens digit change?

## Possibles for $22 \times 23$

- You multiply  $22 \times 23$ . How do you know the answer is closer to 400 than 900 without getting the answer?
- How could knowing that  $22 \times 22 = 484$  help you figure out  $22 \times 23$ ?

Possibles for  $\frac{7}{3} \times \frac{3}{5}$

- Without getting the answer, how do you know  $\frac{7}{3} \times \frac{3}{5}$  is more than  $\frac{3}{5}$  but less than  $\frac{7}{3}$ ?
- Without getting the answer, how do you know that  $\frac{7}{3} \times \frac{3}{5}$  is more than 1?

Possibles for line  
through  $(2,3)$  and  
 $(5,-7)$

- Without calculating the slope, how do you know it is negative?
- Without figuring out the equation of the line, name some points you know are NOT on the line.



Let's look at more  
questions

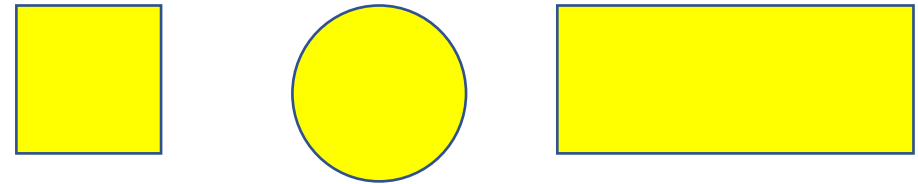
- A number is closer to 10 than 5. What could it be?  
Are there more choices?

K-2



K-3

- DO IT:
- Which is the square?



- UNDERSTAND IT:
- In what ways are these two shapes alike?

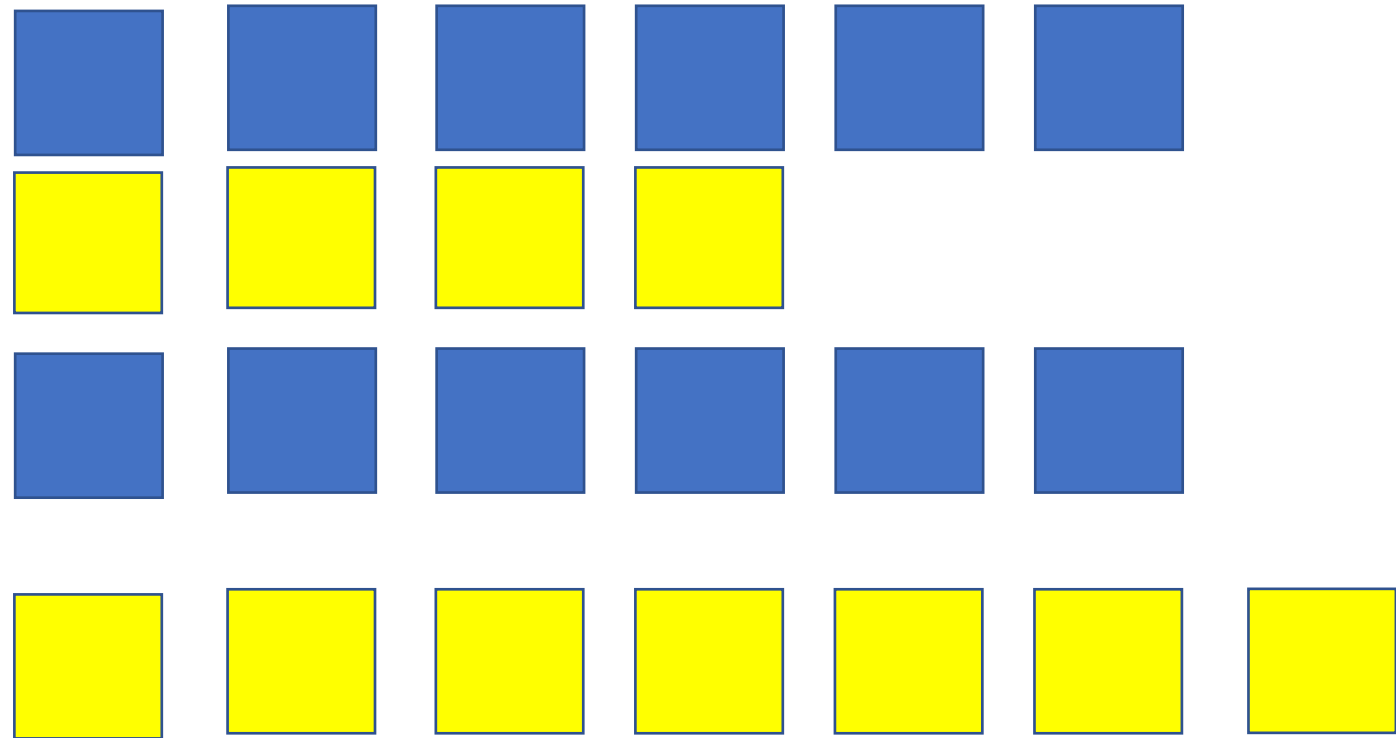


K-3

- DO IT:
- What is  $5 + 5$ ?
  
- UNDERSTAND IT:
- WITHOUT MENTIONING ANY ANSWERS, how do you know that  $6 + 6$  is more than  $4 + 7$ ?

- WITHOUT MENTIONING ANY ANSWERS, how do you know that  $6 + 6$  is more than  $4 + 7$ ?

K-3



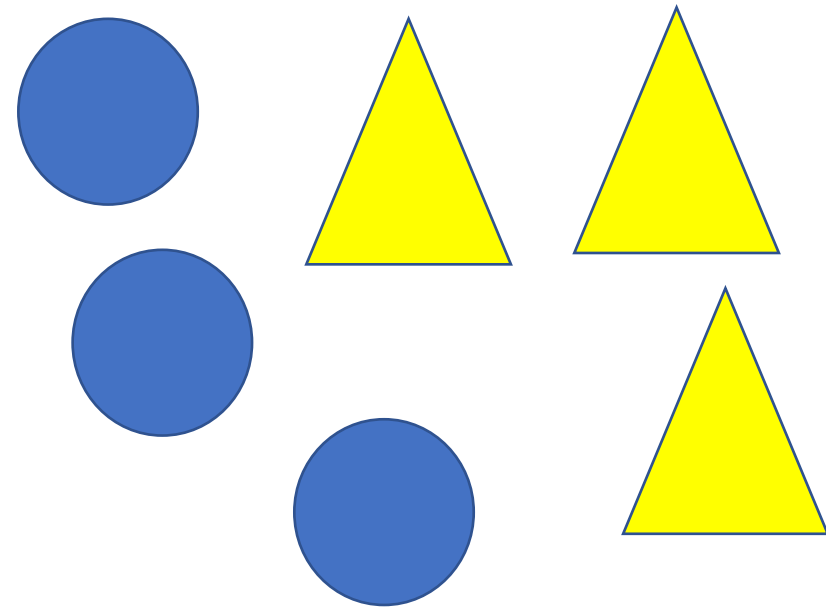
K-3

- DO IT:
  - Measure how many paper clips long.
- 

- UNDERSTAND IT:
- Do you know which is longer- a line that is 4 big paper clips long or a line that is 6 regular paper clips long? Explain.

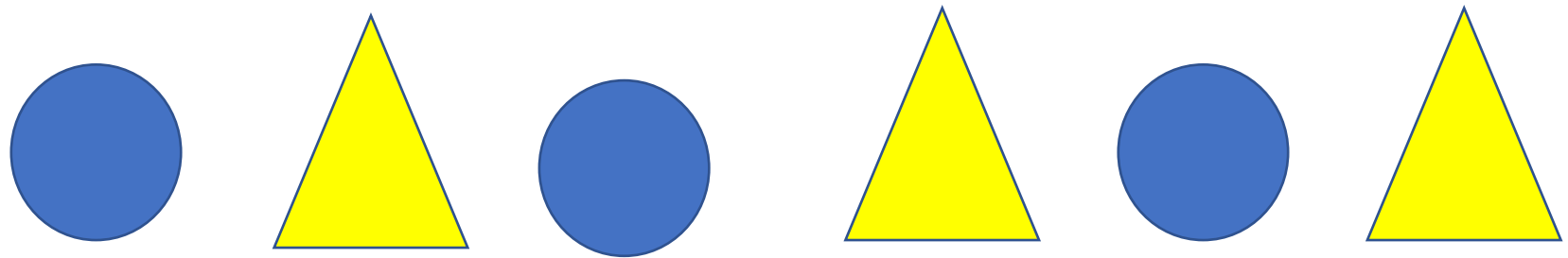
K-3

- DO IT:
- Show an AB pattern with these counters.



- DO IT:
- Show an AB pattern with these counters.

K-3





K-3

- UNDERSTAND IT:
- Why would someone say Patterns 1 and 2 are more alike? Why would they say Patterns 1 and 3 are more alike?

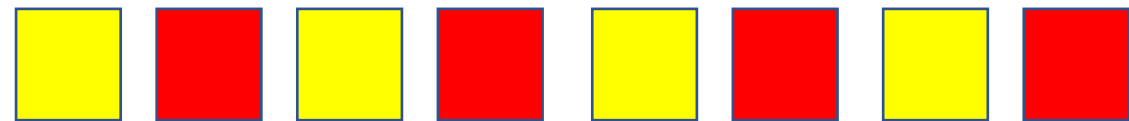
- Pattern 1



- Pattern 2



- Pattern 3



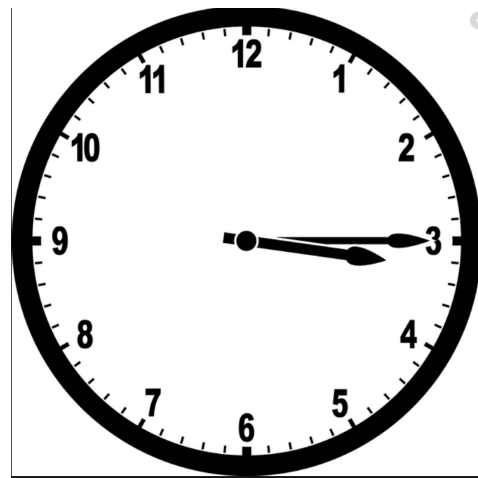
K-3

- DO IT:
- Decompose 42 into tens and ones.
  
- UNDERSTAND IT:
- How many ways can you decompose 42 into tens and ones?

K-3

- How many ways can you decompose 42 into tens and ones?
- 4 tens and 2 ones
- 3 tens and 12 ones
- 2 tens and 22 ones
- 1 ten and 32 ones
- 0 tens and 42 ones

K-3



- DO IT:
- What time does this show?
  
- UNDERSTAND IT:
- The short hand of a clock is between 1 and 2. What time could it be?

K-3

- OR
- The long hand of a clock is between 1 and 2. What time could it be?
- 3:05
- 4:08
- 12:07
- 6:05

Grades 4 - 6

- DO IT:
- Which is more:  
 $\frac{2}{3}$  or  $\frac{1}{5}$ ?
- UNDERSTAND IT:
- Draw a picture that shows why  $\frac{2}{3}$  is more than  $\frac{1}{5}$ .

## Grades 4 - 6

- DO IT:
- Write 47 002 in expanded notation.
  
- UNDERSTAND IT:
- You write a number more than 1000 in expanded notation and it only has two parts. What could it be?

## Grades 4 - 6

- DO IT:
- How many centimetres is 4.2 m?
  
- UNDERSTAND IT:
- When you describe a measurement with different units, you multiply by 1000. What could the original unit and final unit be?



## Grades 4 - 6

- DO IT:
- List the first 5 prime numbers more than 20.
  
- UNDERSTAND IT:
- How could you prove that 47 is a prime number?

- E.g. Make rectangles with 47 square tiles.

Grades 4 - 6

Grades 4 - 6

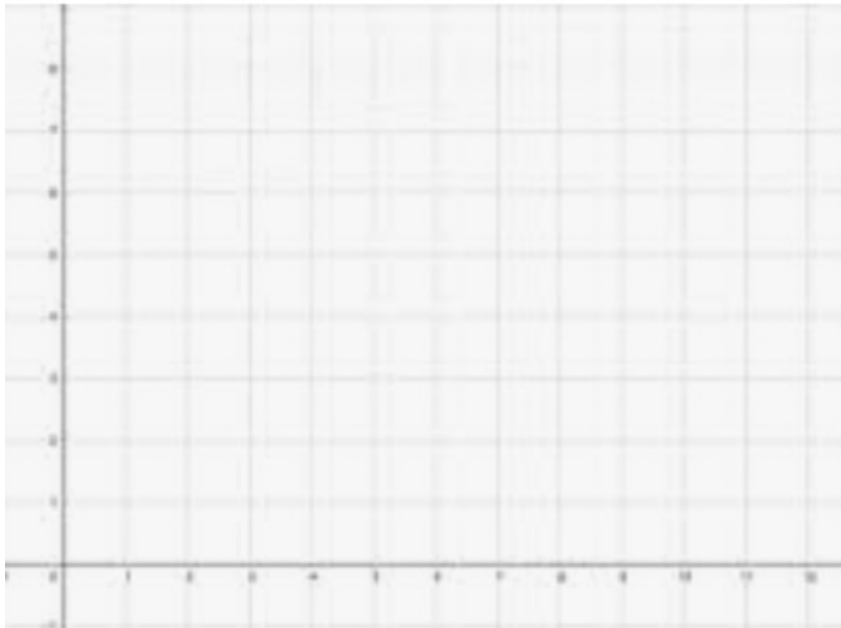
- Use the Sieve of Eratosthenes

1?	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Numbers that divide by 2 in GREEN

Numbers that divide by 3 in BLUE

## Grades 4 - 6



- DO IT:
- Plot the point  $(3,4)$  and  $(5,7)$  on a coordinate grid.
- UNDERSTAND IT:
- The point  $([], *)$  is very far to the right of  $(0,0)$  but not that high up. What could the coordinates be?

## Grades 4 - 6

- DO IT:
- What is  $3425 \div 100$ ?
  
- UNDERSTAND IT:
- Why are the digits the same when you divide by 100?

Grades 7 - 9

- DO IT:
- What is  $-4 - (-9)$ ?
  
- UNDERSTAND IT:
- Draw a picture to show why  $4 - (-9)$  must be 13.

## Grades 7 - 9

- DO IT:
- What is  $8 - (-12)$ ?
  
- UNDERSTAND IT:
- You subtract two integers and the answer is negative. Are the integers negative or positive or could they be either? Explain without using examples.

## Grades 7 - 9

- DO IT:
- What is  $5/8 \div 7/6$ ?
  
- UNDERSTAND IT:
- Draw a picture to show why  $3/5 \div 1/5 = 3$ .



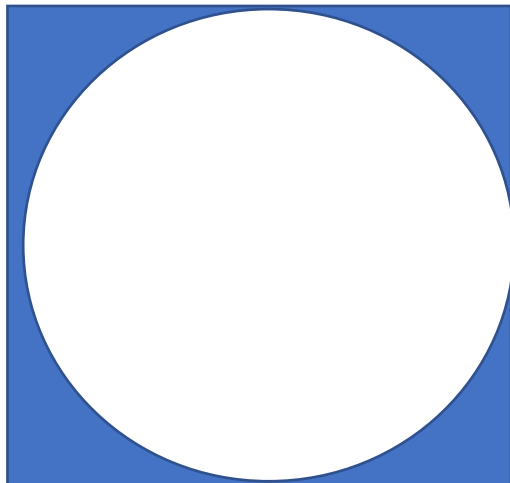
## Grades 7 - 9

- DO IT:
- What is  $5/8 \div 7/6$ ?
  
- UNDERSTAND IT:
- $5/8 \div a/b < 1$ .
- What do you know, for certain, about  $a/b$ ?

## Grades 7 - 9

- DO IT: Calculate:  $-\frac{2}{3} + \frac{5}{2} \cdot \frac{3}{2} - \frac{2}{5}$
- UNDERSTAND IT: A certain expression involving rational numbers can be calculated correctly from left to right, but another expression cannot be calculated that way. Show examples of each.

Grades 7 -9



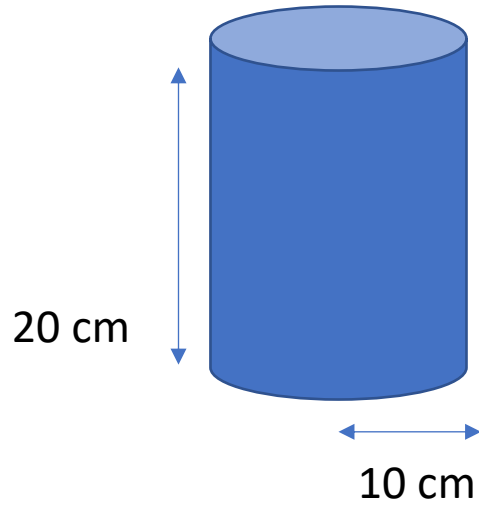
- DO IT:
- What is the area of a circle with a diameter of 10 cm?
  
- UNDERSTAND IT:
- Draw a picture to show why the area of a circle with diameter 10 cm has to be about  $\frac{3}{4}$  of  $100 \text{ cm}^2$ .

## Grades 7 -9

- DO IT:
- What is  $5/8 \times 7/6$ ?
  
- UNDERSTAND IT:
- I want to multiply two fractions and the answer has to be just a little bit less than each of them. What could I multiply?

## Grades 7 -9

- DO IT:
- Solve  $100x + 6 = 87x + 2$
  
- UNDERSTAND IT:
- WITHOUT SOLVING, tell why the solution to  $100x + 6 = 87x + 2$  HAS TO be negative.

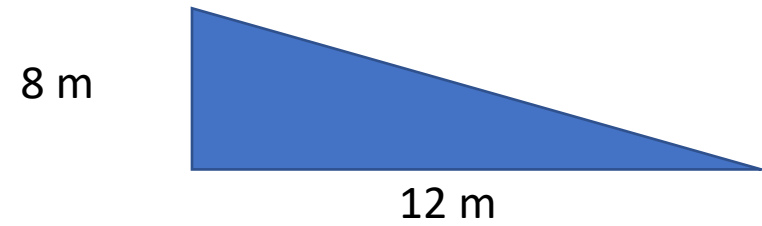


Grades 7 - 9

- DO IT:
  - What is the volume of this cylinder?
- 
- UNDERSTAND IT:
  - A tall cylinder and a short one have the same volume. How is that possible?

## Grades 7 - 9

- DO IT: How long is the hypotenuse?



- UNDERSTAND IT: In a right triangle, two sides are more than 20 cm long but pretty close together in length. What else do you know about the triangle?

## Grades 9-12

- DO IT: What is the vertex of the parabola  $y = 5x^2 - 4x + 8$ ?
- UNDERSTAND IT: Why does knowing the vertex of a parabola not give you enough information to know its equation?



## Grades 9-12

- DO IT: Write the first 20 terms of the pattern that follows this rule: If the term is even, the term value is  $4n$ . If the term is odd, the term value is  $3n - 2$ .
- UNDERSTAND IT: In what mathematical situations might you use an if-then statement

Grades 9-12

- DO IT: What is the number of combinations  ${}_5C_2$ ?
- UNDERSTAND IT: WITHOUT ACTUALLY CALCULATING, tell why it makes sense that  ${}_{10}C_2$  is NOT double  ${}_5C_2$ .

Grades 9-12

- DO IT: What is the number of combinations  ${}_5C_2$ ?
- UNDERSTAND IT: Why do  ${}_5C_2$  and  ${}_5C_3$  HAVE TO be the same value?

## Grades 9-12

- DO IT: Solve:
- $2x + 3y \geq 20$
- $5x - 7 < 15$
  
- UNDERSTAND IT: Without solving the inequalities, why does it make sense that the solutions are more likely to be in Quadrant 2 than Quadrant 4?

Grades 9-12

- DO IT: What is  $\log_{10} 231$ ?
- UNDERSTAND IT:  
WITHOUT CALCULATING,  
tell why  $\log_{10} 231$  has to be  
less than  $\log_8 231$ .

## Grades 9-12

- DO IT: Graph  $y = 5 \cdot 2^x$ .
- UNDERSTAND IT: Predict how these two graphs will be different and why:
  - $y = 5 \cdot 2^x$
  - $y = 5 \cdot 4^x$

- Try to change one of these questions to focus on understanding.

Let's try again

What is  $53 - 17$ ?

Write 24 140 in expanded form.

- What is  $\frac{3}{5} \div \frac{5}{6}$ ?
- What is the tangent of  $48^\circ$ ?



Possibles for  $53 - 17$

- Without getting answers, how do you know  $53 - 17$  has to be more than  $46 - 25$ ?
- Why is  $53 - 17$  more than  $53 - 20$ ?

Possibles for Write  
24 140 in expanded  
form.

- You write a number in expanded form and need four pieces. How big or small might the number be?
- Why might a 5 digit number only have 2 terms in its expanded form?

Possibles for  $\frac{3}{5} \div \frac{5}{6}$

- Without getting an answer, tell why  $\frac{3}{5} \div \frac{5}{6}$  has to be more than  $\frac{3}{5}$ ?
- Without getting an answer, tell how the answers to  $\frac{3}{5} \div \frac{1}{6}$  and  $\frac{3}{5} \div \frac{5}{6}$  compare.

# Possibles for tangent of $48^\circ$

- How do you know the tangent for  $48^\circ$  is more than 1 without actually looking it up?
- If you double an angle's size, do you usually double its tangent?

In general

- Do we still want to ask knowledge questions?
- Of course.
- But what do you think should be the proportion of knowledge to understanding?